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Extending the NAWM for the import content of exports

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ABSTRACT

In this document, we set out the details for extending the New Area-Wide Model (NAWM; cf. Christoffel, Coenen and Warne, 2008) with a non-zero import content of exports. We first describe the technology used by the intermediate-good firms for producing their differentiated outputs sold abroad. We then formulate the modified market clearing conditions as well as the aggregate resource constraint for the extended model. Finally, we outline the computation of the modified steady state.

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The Intermediate-Good Firms' Optimal Choice of Inputs

By Günter Coenen and Igor Vetlov*

In this note, we describe the technology used by the intermediate-good firms for producing their differentiated outputs. In extension of the baseline version of the New Area-Wide Model (NAWM), we allow for a non-zero import content of the intermediate goods that are sold abroad. We then characterise the firms' optimal choices of capital and labour inputs, and present the implied marginal cost schedules. We finally report the log-linearised versions of all expressions.

Technology

There is a continuum of monopolistically competitive firms indexed by $f \in [0,1]$, each of which produces a differentiated intermediate good $Y_{f,t}$ with an increasing-returns-to-scale Cobb-Douglas technology that is subject to fixed costs of production, $z_t\,\psi$,

$$Y_{f,t} \;=\; \max\left[\varepsilon_t\,(K^s_{f,t})^{\alpha}\,(z_t\,N_{f,t})^{1-\alpha} - z_t\,\psi,\, 0\right], \tag{1}$$

utilising as inputs homogenous capital services, $K^s_{f,t}$, that are rent from households in fully competitive markets, and an index of differentiated labour services, $N_{f,t}$, which combines household-specific varieties of labour that are supplied in monopolistically competitive markets. The variable ε_t represents a transitory technology shock that affects total-factor productivity, while the variable z_t denotes a permanent technology shock affecting the productivity of labour. Both shocks, and the fixed cost of production, are assumed to be identical across firms. The fixed cost is scaled by the permanent technology shock to guarantee that the fixed cost as a fraction of output do not vanish as output grows.[1]

The permanent technology shock, which introduces a unit root in the firms' output, is assumed to evolve according to the following serially correlated process,

$$g_{z,t} \;=\; (1-\rho_{g_z})\,g_z + \rho_{g_z}\,g_{z,t-1} + \eta^{g_z}_t, \tag{2}$$

where $g_{z,t} = z_t/z_{t-1}$ represents the (gross) rate of labour-augmenting productivity growth with steady-state value g_z.

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[1]The parameter ψ will be chosen to ensure zero profits in steady state. This in turn guarantees that there is no incentive for other firms to enter the market in the long run.

The differentiated intermediate good $Y_{f,t}$ can be sold to domestic producers of final goods or it can be combined with imported foreign intermediate goods, $IM^X_{f,t}$, and then be sold abroad; that is, exported. The import content of the exported intermediate good is modelled by assuming that the production of the exported good $X_{f,t}$ features the following CES production technology:

$$X_{f,t} = \left((\nu_{X,t})^{\frac{1}{\mu_X}} \left(H^X_{f,t} \right)^{\frac{\mu_X - 1}{\mu_X}} + (1-\nu_{X,t})^{\frac{1}{\mu_X}} \left(IM^X_{f,t} \right)^{\frac{\mu_X - 1}{\mu_X}} \right)^{\frac{\mu_X}{\mu_X - 1}}, \tag{3}$$

where $H^X_{f,t}$ and $IM^X_{f,t}$ are respectively the domestic and the foreign intermediate-good inputs used in the production, the latter input being given by the index

$$IM^X_{f,t} = \left(\int_0^1 \left(IM^{X,f}_{f^*,t} \right)^{\frac{1}{\varphi^*_t}} df^* \right)^{\varphi^*_t}, \tag{4}$$

where the possibly time-varying parameter φ^*_t is inversely related to the intratemporal elasticity of substitution between the differentiated goods supplied by foreign exporters, with $\theta^*_t = \varphi^*_t / (\varphi^*_t - 1) > 1$.

In the production technology (3), the parameter $\mu_X > 1$ is the intratemporal elasticity of substitution between the domestic and the foreign intermediate-good inputs; and the possibly time-varying parameter $\nu_{X,t} \in [0,1]$ measures the home bias in the production of the exported good.

Inputs and Marginal Costs

Taking the rental cost of capital $R_{K,t}$ and the aggregate wage index W_t as given, the intermediate-good firms optimal demand for capital and labour services must solve the problem of minimising total input cost $R_{K,t}\, K_{f,t} + (1+\tau^{W_f}_t)\, W_t\, N_{f,t}$ subject to the technology constraint (1). Here, $\tau^{W_f}_t$ denotes the payroll tax rate levied on wage payments (representing the firms' contribution to social security).

Defining as $MC_{f,t}$ the Lagrange multiplier associated with the technology constraint (1), the first-order conditions of the firms' cost minimisation problem with respect to capital and labour inputs are given, respectively, by

$$\alpha \frac{Y_{f,t} + z_t\, \psi}{K^s_{f,t}} MC_{f,t} = R_{K,t}, \tag{5}$$

$$(1-\alpha)\,\frac{Y_{f,t}+z_t\psi}{N_{f,t}}\,MC_{f,t} \;=\; (1+\tau_t^{W_f})\,W_t, \tag{6}$$

or, more compactly,

$$\frac{\alpha}{1-\alpha}\,\frac{K^s_{f,t}}{N_{f,t}} \;=\; \frac{R_{K,t}}{(1+\tau_t^{W_f})\,W_t}. \tag{7}$$

The Lagrange multiplier $MC_{f,t}$ measures the shadow price of varying the use of capital and labour services; that is, nominal marginal cost. We note that, since all firms f face the same input prices and since they all have access to the same production technology, nominal marginal cost $MC_{f,t}$ are identical across firms; that is, $MC_{f,t} = MC_t$ with

$$MC_t \;=\; \frac{1}{\varepsilon_t\,\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\,(R_{K,t})^{\alpha}((1+\tau_t^{W_f})\,W_t)^{1-\alpha}. \tag{8}$$

Similarly, to determine the optimal demand for the domestic and foreign inputs in the production of exports, an intermediate-good producer must solve the problem of minimising total input cost $MC_t H^X_{f,t} + P_{IM,t} IM^X_{f,t}$ subject to the technology constraint (3), taking the prices of the inputs, MC_t and $P_{IM,t}$, as given.

Defining as $MC^X_{f,t}$ the Lagrange multiplier associated with the technology constraint (3), the first-order conditions of the firm's cost minimisation problem with respect to domestic and foreign inputs are given by

$$MC^X_{f,t}\,(\nu_{X,t})^{\frac{1}{\mu_X}}\left(\frac{X_{f,t}}{H^X_{f,t}}\right)^{\frac{1}{\mu_X}} \;=\; MC_t, \tag{9}$$

$$MC^X_{f,t}\,(1-\nu_{X,t})^{\frac{1}{\mu_X}}\left(\frac{X_{f,t}}{IM^X_{f,t}}\right)^{\frac{1}{\mu_X}} \;=\; P_{IM,t}. \tag{10}$$

The above optimality conditions determine firm's f demand for domestic and imported intermediate goods:

$$H^X_{f,t} \;=\; \nu_{X,t}\left(\frac{MC_t}{MC^X_{f,t}}\right)^{-\mu_X} X_{f,t}, \tag{11}$$

$$IM^X_{f,t} \;=\; (1-\nu_{X,t})\left(\frac{P_{IM,t}}{MC^X_{f,t}}\right)^{-\mu_X} X_{f,t}. \tag{12}$$

Substituting (11) and (12) into the production technology (3), we can express the Lagrange multiplier, or nominal marginal cost, $MC^X_{f,t}$, in terms of the given input prices:

$$MC^X_{f,t} \;=\; \left(\nu_{X,t}\,(MC_t)^{1-\mu_X} + (1-\nu_{X,t})(P_{IM,t})^{1-\mu_X}\right)^{\frac{1}{1-\mu_X}}. \tag{13}$$

Again, since all firms f face the same input prices and the same production technology, nominal marginal cost $MC_{f,t}^X$ will be identical across firms; that is, $MC_{f,t}^X = MC_t^X$.

Finally, recalling that $IM_{f,t}^X$ represents an index of imported differentiated intermediate goods, the demand by firm f for the imported good $IM_{f^*,t}$ is given by

$$IM_{f^*,t}^{X,f} = \left(\frac{P_{IM,f^*,t}}{P_{IM,t}}\right)^{-\frac{\varphi_t^*}{\varphi_t^*-1}} IM_{f,t}^X, \tag{14}$$

where $P_{IM,f^*,t}$ and $P_{IM,t}$ denote the price of the imported differentiated good f^* and the aggregate import price index, respectively. Hence, the aggregate demand by domestic producers for the imported good is given by

$$IM_{f^*,t}^X = \int_0^1 IM_{f^*,t}^{X,f}\, df = \left(\frac{P_{IM,f^*,t}}{P_{IM,t}}\right)^{-\frac{\varphi_t^*}{\varphi_t^*-1}} IM_t^X, \tag{15}$$

where $IM_t^X = \int_0^1 IM_{f,t}^X\, df$.

The Log-Linearised Equations

We now present the log-linearised versions of production technologies (1) and (3), the combined first-order condition (7) and the optimal demand relations (11) and (12), which jointly characterise the firms' optimal choice of inputs, and the marginal cost schedules (8) and (13). We first transform all variables into stationary quantities, and then proceed with the log-linearisation of the resulting expressions around the deterministic steady state. In so doing, the firm-specific index f can be dropped because all firms choose identical inputs in equilibrium.

Transformation of Variables

Because of the assumed unit-root technology of the intermediate-good firms, output and factor inputs contain a real stochastic trend. Similarly, since we allow for a unit-root in nominal variables, the wage index and the rental rate of capital contain a nominal stochastic trend. To render the relevant variables stationary, we scale all variables that contain a real trend with the level of productivity z_t, while we scale all variables that contain a nominal trend with the price of the consumption good $P_{C,t}$.

In order to simplify notation, we introduce the convention that all scaled variables are represented by lower-case letters, rather than by the upper-case letters used so far. For example, we use $y_t = Y_t/z_t$ to denote the stationary level of output, while we use $r_{K,t} = R_{K,t}/P_{C,t}$ to represent the rental rate of capital relative to the price of the consumption good. Note that, since the quantity of labour services is assumed to be stationary, the wage index is non-stationary reflecting productivity trends; and thus the latter needs to be scaled with z_t to become stationary. Accordingly we define $w_t = W_t/(z_t\, P_{C,t})$. Furthermore, as an exception to be motivated elsewhere, we define $k_t^s = K_t^s/z_{t-1}$.

With these conventions, and assuming that domestic production is positive, the stationarity-inducing transformation of the production technologies results in:

$$y_t \;=\; \varepsilon_t\,(g_{z,t}^{-1}\,k_t^s)^{\alpha}\,N_t^{1-\alpha} - \psi, \tag{16}$$

$$x_t \;=\; \left((\nu_{X,t})^{\frac{1}{\mu_X}}\left(h_t^X\right)^{\frac{\mu_X-1}{\mu_X}} + (1-\nu_{X,t})^{\frac{1}{\mu_X}}\left(im_t^X\right)^{\frac{\mu_X-1}{\mu_X}}\right)^{\frac{\mu_X}{\mu_X-1}}. \tag{17}$$

Similarly, the transformation of the combined first-order condition (7) yields:

$$\frac{\alpha}{1-\alpha}\,\frac{g_{z,t}\,N_t}{k_t^s} \;=\; \frac{r_{K,t}}{(1+\tau_t^{W_f})\,w_t}, \tag{18}$$

while using the optimal foreign demand schedules for the differentiated domestic intermediate goods (11) and (12), and integrating over the continuum of domestic intermediate-good producers, the transformation of variables yields:

$$h_t^X \;=\; \nu_{X,t}\,s_{X,t}\left(\frac{mc_t}{mc_t^X}\right)^{-\mu_X} x_t, \tag{19}$$

$$im_t^X \;=\; (1-\nu_{X,t})\,s_{X,t}\left(\frac{p_{IM,t}}{mc_t^X}\right)^{-\mu_X} x_t, \tag{20}$$

where the variable

$$s_{X,t} \;=\; \int_0^1 \left(\frac{P_{X,f,t}}{P_{X,t}}\right)^{-\frac{\varphi_t^X}{\varphi_t^X-1}} df \tag{21}$$

measures the degree of price dispersion across the differentiated goods f sold abroad.

Finally, the transformed marginal cost schedules are given by:

$$mc_t \;=\; \frac{1}{\varepsilon_t\,\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\,(r_{K,t})^{\alpha}((1+\tau_t^{W_f})\,w_t)^{1-\alpha}, \tag{22}$$

$$mc_t^X = \left(\nu_{X,t}(mc_t)^{1-\mu_X} + (1-\nu_{X,t})(p_{IM,t})^{1-\mu_X}\right)^{\frac{1}{1-\mu_X}}. \tag{23}$$

Log-Linearisation around the Deterministic Steady State

Indicating the percentage-point deviation of a variable from its steady-state value by a hat ('$\widehat{\ }$') and implicitly defining the steady-state value of a variable by dropping the time subscript, we obtain the following log-linearised expressions for the production technologies (16) and (17), the relations (18), (19) and (20) characterising optimal factor inputs, and the marginal cost schedules (22) and (23):

$$\widehat{y}_t = (1+\psi\, y^{-1})\left(\widehat{\varepsilon}_t + \alpha\,(\widehat{k}_t^s - \widehat{g}_{z,t}) + (1-\alpha)\,\widehat{N}_t\right), \tag{24}$$

$$\begin{aligned}\widehat{x}_t = {} & (\nu_X)^{\frac{1}{\mu_X}}\left(\frac{h^X}{x}\right)^{\frac{\mu_X-1}{\mu_X}}\widehat{h}_t^X + (1-\nu_X)^{\frac{1}{\mu_X}}\left(\frac{im^X}{x}\right)^{\frac{\mu_X-1}{\mu_X}}\widehat{im}_t^X \\ & + \frac{1}{\mu_X-1}\left((\nu_X)^{\frac{1}{\mu_X}}\left(\frac{h^X}{x}\right)^{\frac{\mu_X-1}{\mu_X}} - \frac{\nu_X}{1-\nu_X}(1-\nu_X)^{\frac{1}{\mu_X}}\left(\frac{im^X}{x}\right)^{\frac{\mu_X-1}{\mu_X}}\right)\frac{1}{\nu_X}\,\widehat{\nu}_{X,t},\end{aligned} \tag{25}$$

$$\widehat{r}_{K,t} = \widehat{g}_{z,t} + \widehat{N}_t + (1+\tau^{W_f})^{-1}\,\widehat{\tau}_t^{W_f} + \widehat{w}_t - \widehat{k}_t^s, \tag{26}$$

$$\widehat{h}_t^X = \widehat{x}_t - \mu_X\,(\widehat{mc}_t - \widehat{mc}_t^X) + \frac{1}{\nu_X}\,\widehat{\nu}_{X,t}, \tag{27}$$

$$\widehat{im}_t^X = \widehat{x}_t - \mu_X(\widehat{p}_{IM,t} - \widehat{mc}_t^X) - \frac{1}{1-\nu_X}\widehat{\nu}_{X,t}, \tag{28}$$

$$\widehat{mc}_t = -\,\widehat{\varepsilon}_t + \alpha\,\widehat{r}_{K,t} + (1-\alpha)\left((1+\tau^{W_f})^{-1}\,\widehat{\tau}_t^{W_f} + \widehat{w}_t\right), \tag{29}$$

$$\begin{aligned}\widehat{mc}_t^X = {} & \nu_X\left(\frac{mc}{mc^X}\right)^{1-\mu_X}\widehat{mc}_t + (1-\nu_X)\left(\frac{p_{IM}}{mc^X}\right)^{1-\mu_X}\widehat{p}_{IM,t} \\ & + \frac{1}{1-\mu_X}\left(\left(\frac{mc}{mc^X}\right)^{1-\mu_X} - \left(\frac{p_{IM}}{mc^X}\right)^{1-\mu_X}\right)\widehat{\nu}_{X,t}.\end{aligned} \tag{30}$$

Market Clearing and Aggregate Resource Constraint

By Günter Coenen and Igor Vetlov*

In this note, we formulate the market clearing conditions and the aggregate resource constraint for an extended version of the New Area-Wide Model (NAWM) which allows for a non-zero important content in the domestic intermediate goods sold abroad.

Market Clearing Conditions

Market Clearing in the Labour Markets

Each household h acts as wage setter in a monopolistically competitive market. Hence, in equilibrium the supply of its differentiated labour service needs to equal intermediate-good firms' demand,

$$N_{h,t} \;=\; \int_0^1 N_{f,t}^h \, df \;=\; N_t^h. \tag{1}$$

Aggregating over the continuum of households h, we have

$$\begin{aligned}\int_0^1 N_{h,t}\, dh &= \int_0^1 N_t^h \, dh \\ &= \int_0^1 \left(\frac{W_{h,t}}{W_t}\right)^{-\frac{\varphi_t^W}{\varphi_t^W - 1}} N_t \, dh \\ &= s_{W,t}\, N_t, \end{aligned} \tag{2}$$

where the variable

$$s_{W,t} \;=\; \int_0^1 \left(\frac{W_{h,t}}{W_t}\right)^{-\frac{\varphi_t^W}{\varphi_t^W - 1}} dh \tag{3}$$

measures the degree of wage dispersion across the differentiated labour services h.

Given the optimal wage-setting strategies for the households, the measure of wage dispersion evolves according to

$$s_{W,t} \;=\; (1-\xi_W)\left(\frac{\tilde{W}_t}{W_t}\right)^{-\frac{\varphi_t^W}{\varphi_t^W - 1}} + \xi_W \left(\frac{W_t}{W_{t-1}}\,\frac{\Pi_{C,t}}{\Pi_{C,t-1}^{\chi_W}\bar{\Pi}_t^{1-\chi_W}}\right)^{-\frac{\varphi_t^W}{\varphi_t^W - 1}} s_{W,t-1}, \tag{4}$$

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where $\tilde{W}_t$ denotes the optimal wage contract chosen by those households that have received permission to reset their wages in period t, and $\Pi_{C,t} = P_{C,t}/P_{C,t-1}$.[1]

As regards the total wage sum paid by firms to the households, we have

$$\begin{aligned} \int_0^1 W_{h,t}\, N_{h,t}\, dh &= N_t \int_0^1 W_{h,t} \left(\frac{W_{h,t}}{W_t}\right)^{-\frac{\varphi_t^W}{\varphi_t^W - 1}} dh \\ &= W_t\, N_t, \end{aligned} \tag{5}$$

where the first equality has been obtained using the aggregate demand for labour services of variety h, while the last equality has been obtained using the properties of the wage index W_t.

Market Clearing in the Intermediate-Good Markets

Each intermediate-good producing firm f acts as price setter in domestic and foreign monopolistically competitive markets. Hence, in equilibrium the supply of its differentiated output needs to equal domestic and foreign demand,

$$Y_{f,t} = H_{f,t} + H_{f,t}^X, \tag{6}$$

where $H_{f,t}^X$ refers to the domestic component of the output sold abroad, which is given by

$$H_{f,t}^X = \nu_{X,t} \left(\frac{MC_t}{MC_t^X}\right)^{-\mu_X} X_{f,t} \tag{7}$$

with

$$X_{f,t} = \left(\frac{P_{X,f,t}}{P_{X,t}}\right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} X_t. \tag{8}$$

Aggregating over the continuum of firms f, we have

$$\begin{aligned} Y_t = \int_0^1 Y_{f,t}\, df &= \int_0^1 H_{f,t}\, df + \int_0^1 H_{f,t}^X\, df \\ &= \int_0^1 \left(\frac{P_{H,f,t}}{P_{H,t}}\right)^{-\frac{\varphi_t^H}{\varphi_t^H - 1}} H_t\, df + \nu_{X,t} \left(\frac{MC_t}{MC_t^X}\right)^{-\mu_X} \int_0^1 \left(\frac{P_{X,f,t}}{P_{X,t}}\right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} X_t\, df \\ &= s_{H,t}\, H_t + \nu_{X,t}\, s_{X,t} \left(\frac{MC_t}{MC_t^X}\right)^{-\mu_X} X_t, \end{aligned} \tag{9}$$

[1] Notice that $s_{W,t}$ is equal to one in steady state and fluctuations in $s_{W,t}$ do vanish in the log-linearised version of the model.

where the variables

$$s_{H,t} = \int_0^1 \left(\frac{P_{H,f,t}}{P_{H,t}} \right)^{-\frac{\varphi_t^H}{\varphi_t^H - 1}} df, \tag{10}$$

$$s_{X,t} = \int_0^1 \left(\frac{P_{X,f,t}}{P_{X,t}} \right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} df \tag{11}$$

measure the degree of price dispersion across the differentiated goods f sold either domestically or abroad.

Given the optimal price-setting strategies for intermediate-good firms, the two measures of price dispersion evolve according to

$$s_{H,t} = (1 - \xi_H) \left(\frac{\tilde{P}_{H,t}}{P_{H,t}} \right)^{-\frac{\varphi_t^H}{\varphi_t^H - 1}} + \xi_H \left(\frac{\Pi_{H,t}}{\Pi_{H,t-1}^{\chi_H} \bar{\Pi}_t^{1-\chi_H}} \right)^{-\frac{\varphi_t^H}{\varphi_t^H - 1}} s_{H,t-1}, \tag{12}$$

$$s_{X,t} = (1 - \xi_X) \left(\frac{\tilde{P}_{X,t}}{P_{X,t}} \right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} + \xi_X \left(\frac{\Pi_{X,t}}{\Pi_{X,t-1}^{\chi_X} \bar{\Pi}_t^{1-\chi_X}} \right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} s_{X,t-1}, \tag{13}$$

where $\tilde{P}_{H,t}$ and $\tilde{P}_{X,t}$ denote the optimal price contracts chosen by those firms that have received permission to reset their prices in their home and foreign markets in period t, and $\Pi_{H,t} = P_{H,t}/P_{H,t-1}$ and $\Pi_{X,t} = P_{X,t}/P_{X,t-1}$.[2]

Similarly, in nominal terms we have

$$\begin{aligned}
P_{Y,t} Y_t &= \int_0^1 P_{H,f,t}\, H_{f,t}\, df + \int_0^1 S_t\, P_{X,f,t}\, X_{f,t}\, df - \int_0^1 P_{IM,t}\, IM_{f,t}^X\, df \qquad (14) \\
&= H_t \int_0^1 P_{H,f,t} \left(\frac{P_{H,f,t}}{P_{H,t}} \right)^{-\frac{\varphi_t^H}{\varphi_t^H - 1}} df + X_t\, S_t \int_0^1 P_{X,f,t} \left(\frac{P_{X,f,t}}{P_{X,t}} \right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} df \\
&\quad - (1 - \nu_{X,t})\, P_{IM,t} \left(\frac{P_{IM,t}}{MC_t^X} \right)^{-\mu_X} X_t \int_0^1 \left(\frac{P_{X,f,t}}{P_{X,t}} \right)^{-\frac{\varphi_t^X}{\varphi_t^X - 1}} df \\
&= P_{H,t}\, H_t + S_t\, P_{X,t}\, X_t - (1 - \nu_{X,t})\, s_{X,t}\, P_{IM,t} \left(\frac{P_{IM,t}}{MC_t^X} \right)^{-\mu_X} X_t \\
&= P_{H,t}\, H_t + \left(S_t\, P_{X,t} - (1 - \nu_{X,t})\, s_{X,t}\, P_{IM,t} \left(\frac{P_{IM,t}}{MC_t^X} \right)^{-\mu_X} \right) X_t,
\end{aligned}$$

[2] Notice that $s_{H,t}$ and $s_{X,t}$ are equal to one in steady state. Furthermore, fluctuations in $s_{H,t}$ and $s_{X,t}$ do vanish in the log-linearised version of the model.

where the second equality has been obtained using the aggregate demand relationships for the domestic intermediate goods sold in home and foreign markets, $H_{f,t}$ and $X_{f,t}$, and the domestic component of the latter $IM^X_{f,t}$, with

$$IM^X_{f,t} = (1-\nu_{X,t})\left(\frac{P_{IM,t}}{MC^X_t}\right)^{-\mu_X} X_{f,t} \tag{15}$$

and

$$X_{f,t} = \left(\frac{P_{X,f,t}}{P_{X,t}}\right)^{-\frac{\varphi^X_t}{\varphi^X_t-1}} X_t, \tag{16}$$

while the third equality has been obtained using the properties of the aggregate price indexes $P_{H,t}$ and $P_{X,t}$ and the previous result on the price dispersion of the intermediate goods sold abroad.

Market Clearing in the Imported-Good Markets

Each foreign exporter f^* acts as price setter for its differentiated output in domestic monopolistically competitive markets. Hence, in equilibrium the supply of its differentiated output needs to equal demand, $IM_{f^*,t}$.

Aggregating over the continuum of firms f^*, we have

$$\begin{aligned} \int_0^1 IM_{f^*,t}\, df^* &= \int_0^1 \left(\frac{P_{IM,f^*,t}}{P_{IM,t}}\right)^{-\frac{\varphi^*_t}{\varphi^*_t-1}} IM_t\, df^* \\ &= s_{IM,t}\, IM_t, \end{aligned} \tag{17}$$

where the variable

$$s_{IM,t} = \int_0^1 \left(\frac{P_{IM,f^*,t}}{P_{IM,t}}\right)^{-\frac{\varphi^*_t}{\varphi^*_t-1}} df^* \tag{18}$$

measures the degree of price dispersion across the differentiated goods f^*.

Given the optimal price-setting strategies for intermediate-good firms, the measure of price dispersion evolves according to

$$s_{IM,t} = (1-\xi^*)\left(\frac{\tilde{P}_{IM,t}}{P_{IM,t}}\right)^{-\frac{\varphi^*_t}{\varphi^*_t-1}} + \xi^*\left(\frac{\Pi_{IM,t}}{\Pi^{\chi^*}_{IM,t-1}\bar{\Pi}^{1-\chi^*}_t}\right)^{-\frac{\varphi^*_t}{\varphi^*_t-1}} s_{IM,t-1}, \tag{19}$$

where $\tilde{P}_{IM,t}$ denotes the optimal price contracts chosen by those importers that have received permission to reset their prices in period t, and $\Pi_{IM,t} = P_{IM,t}/P_{IM,t-1}$.[3]

[3]Like in the case of the domestic intermediate-good producers, $s_{IM,t}$ is equal to one in steady state and fluctuations in $s_{IM,t}$ do vanish in the log-linearised version of the model.

Market Clearing in the Final-Good Markets

Market clearing in the fully competitive final-good markets implies:

$$Q_t^C = C_t, \tag{20}$$

$$Q_t^I = I_t + \Gamma_u(u_t)\, K_t, \tag{21}$$

$$Q_t^G = G_t. \tag{22}$$

Market Clearing in the Capital Market and Distribution of Profits

Market clearing in the rental market for capital services implies that the effective utilisation of capital by households satisfies

$$u_t\, K_t = \int_0^1 K_{f,t}^s\, df = K_t^s. \tag{23}$$

As to the distribution of profits to households, we have

$$\begin{aligned} D_t &= \int_0^1 D_{H,f,t}\, df + \int_0^1 D_{X,f,t}\, df & (24)\\ &= P_{H,t}\, H_t + S_t\, P_{X,t}\, X_t - MC_t\, \left(s_{H,t}\, H_t + \psi\, z_t\right) - MC_t^X\, \left(s_{X,t}\, X_t\right). & (25) \end{aligned}$$

or, written as profit share,

$$s_{D,t} = \frac{D_t}{P_{Y,t}\, Y_t} = \frac{P_{H,t}}{P_{Y,t}} \frac{H_t}{Y_t} + \frac{S_t\, P_{X,t}}{P_{Y,t}} \frac{X_t}{Y_t} - \frac{MC_t}{P_{Y,t}} \frac{s_{H,t}\, H_t + \psi\, z_t}{Y_t} - \frac{MC_t^X}{P_{Y,t}} \frac{s_{X,t}\, X_t}{Y_t}. \tag{26}$$

Market Clearing in the Domestic Government Bond Market

The equilibrium holdings of domestic government bonds evolve over time according to the fiscal authority's budget constraint, reflecting the fiscal authority's need to issue debt in order to finance its deficit.

Market Clearing in the Market for Internationally Traded Bonds

At a given point in time t, the supply of internationally traded foreign bonds is fully elastic, while the holdings of foreign bonds are zero in steady state reflecting the existence of an financial intermediation premium.

Log-Linearisation around the Steady State

Transformation and log-linearisation of the relevant market clearing conditions (9), (20), (21), (22) and (23) yields:[4]

$$\widehat{y}_t = \frac{h}{y}\,\widehat{h}_t + \frac{h^X}{y}\,\widehat{h}^X_t \tag{27}$$

$$\widehat{q}^C_t = \widehat{c}_t, \tag{28}$$

$$\widehat{q}^I_t = \widehat{i}_t + r_K\,p_I^{-1}\,g_z^{-1}\,\frac{k}{q^I}\,\widehat{u}_t, \tag{29}$$

$$\widehat{q}^G_t = \widehat{g}_t, \tag{30}$$

and

$$\widehat{u}_t + \widehat{k}_t = \widehat{k}^s_t, \tag{31}$$

where we recall that $\Gamma'_u(1) = \gamma_{u,1} = r_K\,p_I^{-1}$.

Similarly, rewriting the profit share in terms of stationary variables,

$$s_{D,t} = \frac{p_{H,t}}{p_{Y,t}}\frac{h_t}{y_t} + s_t\,p_{X,t}\,\frac{x_t}{y_t} - \frac{mc_t}{p_{Y,t}}\,\frac{s_{H,t}\,h_t + \psi}{y_t} - \frac{mc^X_t}{p_{Y,t}}\,\frac{s_{X,t}\,x_t}{y_t}, \tag{32}$$

we obtain the following log-linearised expression for the profit share:

$$\begin{aligned}
\widehat{s}_{D,t} = {} & \frac{p_H}{p_Y}\frac{h}{y}\,(\widehat{p}_{H,t} + \widehat{h}_t - \widehat{p}_{Y,t} - \widehat{y}_t) + s\,p_X\,\frac{x}{y}\,(\widehat{s}_t + \widehat{p}_{X,t} + \widehat{x}_t - \widehat{y}_t) \\
& - \frac{mc}{p_Y}\frac{h}{y}\,(\widehat{mc}_t + \widehat{h}_t - \widehat{p}_{Y,t} - \widehat{y}_t) - \frac{mc}{p_Y}\frac{\psi}{y}\,(\widehat{mc}_t - \widehat{p}_{Y,t} - \widehat{y}_t) \\
& - \frac{mc^X}{p_Y}\,\frac{x}{y}\,(\widehat{mc}^X_t + \widehat{x}_t - \widehat{p}_{Y,t} - \widehat{y}_t).
\end{aligned} \tag{33}$$

Aggregate Resource Constraint

The market clearing conditions, jointly with the budget constraints of the households and the fiscal authority, imply the following aggregate resource constraint:

$$\begin{aligned}
P_{Y,t}\,Y_t &= P_{H,t}\,H_t + S_t\;P_{X,t}\,X_t - P_{IM,t}\,IM^X_t \qquad (34)\\
&= P_{C,t}\,C_t + P_{I,t}\,(I_t + \Gamma_u(u_t)\,K_t) + \;P_{G,t}\,G_t + S_t\,P_{X,t}\,X_t \\
&\quad - P_{IM,t}\left(IM^C_t\,\frac{1-\Gamma_{IM^C}(IM^C_t/Q^C_t)}{\Gamma^\dagger_{IM^C}(IM^C_t/Q^C_t)} + IM^I_t\,\frac{1-\Gamma_{IM^I}(IM^I_t/Q^I_t)}{\Gamma^\dagger_{IM^I}(IM^I_t/Q^I_t)} + IM^X_t \right),
\end{aligned}$$

where

$$IM^X_t = \int_0^1 IM^X_{f,t}\,df, \tag{35}$$

[4]For details, see, e.g., Christoffel and Coenen, 2006a, "The Households' Optimal Choice of Allocations", mimeo, ECB.

with $IM^X_{f,t}$ being determined by equations (15) and (16).

Appropriate transformations and subsequent log-linearisation of the aggregate resource constraint yields

$$\begin{aligned}\widehat{p}_{Y,t} + \widehat{y}_t \quad = \quad & \frac{P_C\, C}{P_Y\, Y}\,(\widehat{p}_{C,t} + \widehat{c}_t) + \frac{P_I\, I}{P_Y\, Y}\,(\widehat{p}_{I,t} + \widehat{i}_t) + \frac{P_I\, K}{P_Y\, Y}\,\gamma_{u,1}\,\widehat{u}_t \\ & + \frac{P_G\, G}{P_Y\, Y}\,(\widehat{p}_{G,t} + \widehat{g}_t) + \frac{S\, P_X\, X}{P_Y\, Y}\,(\widehat{s}_t + \widehat{p}_{Y,t} + \widehat{p}_{X,t} + \widehat{x}_t) \\ & - \frac{P_{IM}\, IM^C}{P_Y\, Y}\,\left(\widehat{p}_{IM,t} + \widehat{im}^C_t + \gamma_{IM^C}\,\left(\widehat{im}^C_t - \widehat{q}^C_t - \widehat{im}^C_{t-1} + \widehat{q}^C_{t-1}\right)\right) \\ & - \frac{P_{IM}\, IM^I}{P_Y\, Y}\,\left(\widehat{p}_{IM,t} + \widehat{im}^I_t + \gamma_{IM^I}\,\left(\widehat{im}^I_t - \widehat{q}^I_t - \widehat{im}^I_{t-1} + \widehat{q}^I_{t-1}\right)\right) \\ & - \frac{P_{IM}\, IM^X}{P_Y\, Y}\,\left(\widehat{p}_{IM,t} + \widehat{im}^X_t\right),\end{aligned} \tag{36}$$

where

$$\widehat{im}^X_t \quad = \quad \widehat{x}_t - \mu_X(\widehat{p}_{IM,t} - \widehat{mc}^X_t) - \frac{1}{1-\nu_X}\widehat{\nu}_{X,t}. \tag{37}$$

Computation of the Steady State

By GÜNTER COENEN AND IGOR VETLOV*

In this note, we outline the computation of the steady state for an extended version of the New Area-Wide Model (NAWM) which allows for a non-zero import content of domestic intermediate goods sold abroad. Our strategy is to reduce the steady-state version of the model to a system of seven equations consisting of relations characterising the equilibrium in the labour, capital and goods markets as well as the system of equilibrium relative prices. For given import prices, these relations allow us to solve simultaneously for the steady-state values of the capital stock, k, consumption purchases, c, hours worked, N, the relative price of the investment good, P_I/P_C, the relative prices of the domestic intermediate goods sold domestically and abroad, P_H/P_C and P_X/P_C, and the relative price of aggregate production, P_Y/P_C, with all prices being expressed relative to that of the consumption good.[1]

The Households' Optimal Allocations

We start by stating several first-order conditions characterising the households' optimal allocations in steady state. Because of the assumed unit-root technology and the unit root in prices, all variables that contain a real trend are scaled by the level of productivity z (with trend growth rate g_z), while all variables that contain a nominal trend are scaled by the price of the consumption good P_C.

The first-order condition characterising the households' optimal purchases of the consumption good then yields:

$$\lambda = \frac{1}{(1 - \kappa\, g_z^{-1})\,(1 + \tau^C)\, c}. \tag{1}$$

Similarly, from the first-order condition characterising the optimal purchases of the investment good we obtain:

$$\frac{P_I}{P_C} = Q. \tag{2}$$

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[1]The computation of the steady state follows closely a solution strategy proposed by Paolo Pesenti.

From the first-order condition characterising the optimal holdings of capital we can derive the following steady-state expression for Q:

$$Q \;=\; \frac{1-\tau^K}{g_z\,\beta^{-1}+\delta-1-\tau^K\,\delta}\,\frac{R_K}{P_C}. \tag{3}$$

Evaluating the first-order condition for the capital stock in steady state and making use of the first-order condition for the optimal utilisation of capital,

$$R_K \;=\; \gamma_{u,1}\,P_I, \tag{4}$$

we can determine the first derivative of the capital adjustment function:

$$\gamma_{u,1} \;=\; \frac{R_K}{P_C}\,\frac{P_C}{P_I} \;=\; \frac{1}{1-\tau^K}\,\left(g_z\,\beta^{-1}+\delta-1-\tau^K\,\delta\right). \tag{5}$$

Finally, we note that the capital accumulation equation can be written as

$$i \;=\; \left(1-\frac{1-\delta}{g_z}\right)\,k. \tag{6}$$

Labour-Market Equilibrium

On the labour-supply side, the first-order condition characterising the households' optimal wage-setting decision yields the following (Lerner) relation:

$$(1-\tau^N-\tau^{W_h})\,\frac{W}{P_C\,z} \;=\; \varphi^W\,\frac{N^\zeta}{\lambda}. \tag{7}$$

Using the first-order condition (1), we can re-write this relation as

$$(1-\tau^N-\tau^{W_h})\,\frac{W}{P_C\,z} \;=\; \varphi^W\,N^\zeta\,(1+\tau^C)\,(1-\kappa\,g_z^{-1})\,c \tag{8}$$

or, alternatively,

$$\frac{W}{P_C\,z} \;=\; \frac{1+\tau^C}{1-\tau^N-\tau^{W_h}}\,(1-\kappa\,g_z^{-1})\,\varphi^W\,N^\zeta\,c. \tag{9}$$

Regarding the characterisation of labour demand, we utilise the combined first-order conditions characterising the intermediate-good firms' optimal choice of inputs:

$$\frac{R_K/P_C}{(1+\tau^{W_f})\,W/(P_C\,z)} \;=\; \frac{\alpha}{(1-\alpha)}\,\frac{N}{k}\,g_z, \tag{10}$$

or, alternatively,

$$\frac{W}{P_C\, z} \;=\; \frac{1-\alpha}{\alpha}\,\frac{k\, N^{-1}}{g_z\,(1+\tau^{W_f})}\,\frac{R_K}{P_C}. \tag{11}$$

Combining (9) and (11), we obtain the following equilibrium relation for the labour market:

$$\frac{1+\tau^C}{1-\tau^N-\tau^{W_h}}\,(1-\kappa\, g_z^{-1})\,\varphi^W\, N^\zeta\, c \;=\; \frac{1-\alpha}{\alpha}\,\frac{g_z^{-1} k\, N^{-1}}{1+\tau^{W_f}}\,\frac{R_K}{P_C}. \tag{12}$$

Re-arranging and using (5), we obtain

$$\begin{aligned} g_z\,\frac{1+\tau^C}{1-\tau^N-\tau^{W_h}}\,(1-\kappa\, g_z^{-1})\,\varphi^W\, N^{\zeta+1}\, c\, k^{-1} \;=&\\ \frac{1-\alpha}{\alpha}\,\frac{1}{1+\tau^{W_f}}\,\frac{1}{1-\tau^K}&\left(g_z\,\beta^{-1}+\delta-1-\tau^K\,\delta\right)\frac{P_I}{P_C}. \end{aligned} \tag{13}$$

We can re-write this expression as follows:

$$g_z\,\frac{\alpha}{1-\alpha}\,\frac{1+\tau^C}{1-\tau^N-\tau^{W_h}}\,(1-\kappa\, g_z^{-1})\,\frac{1+\tau^{W_f}}{\gamma_{u,1}}\,\varphi^W\, N^{\zeta+1}\, c\, k^{-1}\,\frac{P_C}{P_I} \;=\; 1. \tag{14}$$

Using the definition

$$\Theta \;=\; g_z\,\frac{\alpha}{1-\alpha}\,\frac{1+\tau^C}{1-\tau^N-\tau^{W_h}}\,(1-\kappa\, g_z^{-1})\,\frac{1+\tau^{W_f}}{\gamma_{u,1}}\,\varphi^W, \tag{15}$$

we finally obtain:

$$\Theta\, N^\zeta\, c\left(\frac{k}{N}\right)^{-1}\frac{P_C}{P_I} \;=\; 1. \tag{16}$$

Capital-Market Equilibrium

Combining the first-order condition characterising the intermediate-good firms' optimal demand for capital services,

$$\alpha\, g_z^{1-\alpha}\left(\frac{N}{k}\right)^{1-\alpha} \;=\; \frac{R_K}{MC}, \tag{17}$$

and the first-order condition characterising the intermediate-good firms' optimal price-setting decision in domestic markets,

$$P_H \;=\; \varphi^H MC, \tag{18}$$

we obtain the following equilibrium condition for the capital market:

$$\alpha \quad = \quad g_z^{-(1-\alpha)} \left(\frac{k}{N}\right)^{1-\alpha} \frac{R_K}{P_C} \varphi^H \frac{P_C}{P_H}. \tag{19}$$

Using (5), we can re-write this equilibrium condition as

$$\alpha \quad = \quad g_z^{-(1-\alpha)} \left(\frac{k}{N}\right)^{1-\alpha} \gamma_{u,1} \varphi^H \frac{P_I}{P_C} \frac{P_C}{P_H}. \tag{20}$$

Goods-Markets Equilibrium

As regards the production of intermediate goods, the following real resource constraint holds in steady state:

$$g_z^{-\alpha} k^\alpha N^{1-\alpha} - \psi \quad = \quad h + h^X, \tag{21}$$

where h^X denotes the amount of domestic intermediate goods used in the production of the intermediate goods sold abroad, which is determined by the following demand equation:

$$h^X \quad = \quad \nu_X \left(\frac{P_H}{P_C} \frac{P_C}{P_X} \frac{\varphi^X}{\varphi^H}\right)^{-\mu_X} x. \tag{22}$$

Here, the parameters φ^H and φ^X denote the intermediate-goods firms' markups over the marginal cost of producing goods for respectively domestic and foreign markets.[2]

Similarly, the demand for imported intermediate goods used in the production of the intermediate good sold abroad is given by

$$im^X \quad = \quad (1-\nu_X) \left(\frac{P_{IM}}{P_C} \frac{P_C}{P_X} \varphi^X\right)^{-\mu_X} x. \tag{23}$$

At the final-goods producers' level, the demand for the bundles of domestic intermediate goods used in the production of the consumption and the investment good, respectively, is given by

$$h^C \quad = \quad \nu_C \left(\frac{P_H}{P_C}\right)^{-\mu_C} c \tag{24}$$

[2]Notice that we have used the fact that the demand for domestic intermediate goods used for producing the intermediate goods sold abroad depends on the ratio of the marginal costs of producing the domestic intermediate goods and the intermediate goods sold abroad, which, in steady state, are given by P_H/φ^H and P_X/φ^X, respectively.

and

$$h^I \quad = \quad \nu_I \left(\frac{P_H}{P_C}\frac{P_C}{P_I}\right)^{-\mu_I} \left(\frac{g_z - 1 + \delta}{g_z}\, k\right), \tag{25}$$

where we have used relation (6).

Similarly, the demand for the bundles of the imported intermediate goods is given by

$$im^C \quad = \quad (1-\nu_C)\left(\frac{P_{IM}}{P_C}\right)^{-\mu_C} c \tag{26}$$

and

$$im^I \quad = \quad (1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{-\mu_I} \left(\frac{g_z - 1 + \delta}{g_z}\, k\right). \tag{27}$$

Taking into account the identity $h = h^C + h^I + h^G$ and using (22) and (24), we can re-write the aggregate resource constraint (21) as

$$h^I \quad = \quad g_z^{-\alpha}\left(\frac{k}{N}\right)^{\alpha} N - \psi - \nu_X \left(\frac{P_H}{P_C}\frac{P_C}{P_X}\frac{\varphi^X}{\varphi^H}\right)^{-\mu_X} x - \nu_C \left(\frac{P_H}{P_C}\right)^{-\mu_C} c - h^G. \tag{28}$$

Substituting (27) and (28) into the investment-good technology,

$$i^{1-\frac{1}{\mu_I}} \quad = \quad \nu_I^{\frac{1}{\mu_I}} \left(h^I\right)^{1-\frac{1}{\mu_I}} + (1-\nu_I)^{\frac{1}{\mu_I}} \left(im^I\right)^{1-\frac{1}{\mu_I}}, \tag{29}$$

and using (6), we then obtain the following expression:

$$\begin{aligned}\left(\frac{g_z - 1 + \delta}{g_z}\, k\right)^{1-\frac{1}{\mu_I}} = \\ \nu_I^{\frac{1}{\nu_I}} \left(g_z^{-\alpha}\left(\frac{k}{N}\right)^{\alpha} N - \psi - \nu_X \left(\frac{P_H}{P_C}\frac{P_C}{P_X}\frac{\varphi^X}{\varphi^H}\right)^{-\mu_X} x - \nu_C \left(\frac{P_H}{P_C}\right)^{-\mu_C} c - h^G\right)^{1-\frac{1}{\mu_I}} \\ + (1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{1-\mu_I} \left(\frac{g_z - 1 + \delta}{g_z}\, k\right)^{1-\frac{1}{\mu_I}},\end{aligned} \tag{30}$$

or, equivalently,

$$\begin{aligned}\left(\frac{g_z - 1 + \delta}{g_z}\, k\right)^{1-\frac{1}{\mu_I}} \left(1 - (1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{1-\mu_I}\right) = \\ \nu_I^{\frac{1}{\mu_I}} \left(g_z^{-\alpha}\left(\frac{k}{N}\right)^{\alpha} N - \psi - \nu_X \left(\frac{P_H}{P_C}\frac{P_C}{P_X}\frac{\varphi^X}{\varphi^H}\right)^{-\mu_X} x - \nu_C \left(\frac{P_H}{P_C}\right)^{-\mu_C} c - h^G\right)^{1-\frac{1}{\mu_I}}.\end{aligned} \tag{31}$$

Equilibrium Relative Prices

Using the price of the consumption good as the numeraire, the relative prices of the domestic intermediate goods sold at home and the investment good are given by

$$1 \;=\; \nu_C \left(\frac{P_H}{P_C}\right)^{1-\mu_C} + (1-\nu_C)\left(\frac{P_{IM}}{P_C}\right)^{1-\mu_C} \tag{32}$$

and

$$\left(\frac{P_I}{P_C}\right)^{1-\mu_I} \;=\; \nu_I \left(\frac{P_H}{P_C}\right)^{1-\mu_I} + (1-\nu_I)\left(\frac{P_{IM}}{P_C}\right)^{1-\mu_I}. \tag{33}$$

Re-arranging equation (32), we obtain:

$$\frac{P_H}{P_C} \;=\; \left(\frac{1-(1-\nu_C)\left(\frac{P_{IM}}{P_C}\right)^{1-\mu_C}}{\nu_C}\right)^{\frac{1}{1-\mu_C}}, \tag{34}$$

and, similarly, re-arranging equation (33) yields:

$$\frac{P_H}{P_C}\frac{P_C}{P_I} \;=\; \left(\frac{1-(1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{1-\mu_I}}{\nu_I}\right)^{\frac{1}{1-\mu_I}}. \tag{35}$$

As the price of the domestic intermediate goods sold abroad is determined as a markup over marginal cost (given by a CES aggregate of the marginal cost of producing the domestic intermediate goods sold at home and the price of the imported intermediate goods), the relative price of the domestic intermediate goods sold abroad is given by

$$\frac{P_X}{P_C} \;=\; \varphi^X \left(\nu_X \left(\frac{P_H}{P_C\varphi^H}\right)^{1-\mu_X} + (1-\nu_X)\left(\frac{P_{IM}}{P_C}\right)^{1-\mu_X}\right)^{\frac{1}{1-\mu_X}}. \tag{36}$$

Finally, the relative price of aggregate production, or output, is determined by the nominal resource constraint:

$$P_Y\, y \;=\; P_H\, h + P_X\, x - P_{IM}\, im^X, \tag{37}$$

or, equivalently, in relative terms:

$$\begin{aligned}\frac{P_Y}{P_C} \;=\; & \left\{\frac{P_H}{P_C}\left(\nu_C\left(\frac{P_H}{P_C}\right)^{-\mu_C} c + \nu_I\left(\frac{P_H}{P_C}\frac{P_C}{P_I}\right)^{-\mu_I}\left(\frac{g_z-1+\delta}{g_z}k\right) + h^G\right)\right. \\ & \left. + \left(\frac{P_X}{P_C} - \frac{P_{IM}}{P_C}(1-\nu_X)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_X}\varphi^X\right)^{-\mu_X}\right) x\right\}\frac{1}{g_z^{-\alpha}k^\alpha N^{1-\alpha}-\psi}.\end{aligned} \tag{38}$$

where we have made use of the identity $h = h^C + h^I + h^G$, together with equations (24) and (25), as well as equation (23).

The Reduced Steady-State Model

Collecting equations (16), (20), (31), (34), (35), (36) and (38), the steady-state model can now be reduced—by conditioning on government consumption, $g = h^G$, exports, x, fixed cost in production, ψ, and the relative price of the bundle of imported goods, P_{IM}/P_C—to a set of seven equations in the unknown steady-state values of the capital stock, k, consumption purchases, c, hours worked, N, the relative price of the investment good, P_I/P_C, the relative prices of domestic intermediate goods sold at home and abroad, P_H/P_C and P_X/P_C, and the relative price of domestic output, P_Y/P_C:[3]

1. Labour-market equilibrium:

$$\Theta\, N^{\zeta}\, c \left(\frac{k}{N}\right)^{-1} \frac{P_C}{P_I} = 1 \tag{39}$$

2. Capital-market equilibrium:

$$\alpha = g_z^{-(1-\alpha)} \left(\frac{k}{N}\right)^{1-\alpha} \gamma_{u,1}\, \varphi^H \frac{P_I}{P_C} \frac{P_C}{P_H} \tag{40}$$

3. Goods-markets equilibrium:

$$\begin{aligned} &\left(\frac{g_z - 1 + \delta}{g_z} k\right)^{1-\frac{1}{\mu_I}} \left(1 - (1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{1-\mu_I}\right) = \\ &\nu_I^{\frac{1}{\mu_I}} \left(g_z^{-\alpha}\left(\frac{k}{N}\right)^{\alpha} N - \psi - \nu_X \left(\frac{P_H}{P_C}\frac{P_C}{P_X}\frac{\varphi^X}{\varphi^H}\right)^{-\mu_X} x - \nu_C \left(\frac{P_H}{P_C}\right)^{-\mu_C} c - h^G\right)^{1-\frac{1}{\mu_I}} \end{aligned} \tag{41}$$

4. Equilibrium relative price of the investment good:

$$\frac{P_C}{P_I} = \frac{P_C}{P_H} \left(\frac{1 - (1-\nu_I)\left(\frac{P_{IM}}{P_C}\frac{P_C}{P_I}\right)^{1-\mu_I}}{\nu_I}\right)^{\frac{1}{1-\mu_I}} \tag{42}$$

5. Equilibrium relative price of intermediate goods sold domestically:

$$\frac{P_H}{P_C} = \left(\frac{1 - (1-\nu_C)\left(\frac{P_{IM}}{P_C}\right)^{1-\mu_C}}{\nu_C}\right)^{\frac{1}{1-\mu_C}} \tag{43}$$

[3] In principle, the system of equation could be further reduced by substituting equations (34) and (36) for P_H/P_C and P_X/P_C. For expositional clarity, however, these equations are retained in the system of equations characterizing the steady-state model.

6. Equilibrium relative price of intermediate goods sold abroad:

$$\frac{P_X}{P_C} = \varphi^X \left(\nu_X \left(\frac{P_H}{P_C \varphi^H} \right)^{1-\mu_X} + (1-\nu_X) \left(\frac{P_{IM}}{P_C} \right)^{1-\mu_X} \right)^{\frac{1}{1-\mu_X}} \tag{44}$$

7. Equilibrium relative price of output:

$$\begin{aligned} \frac{P_Y}{P_C} = & \left\{ \frac{P_H}{P_C} \left(\nu_C \left(\frac{P_H}{P_C} \right)^{-\mu_C} c + \nu_I \left(\frac{P_H}{P_C} \frac{P_C}{P_I} \right)^{-\mu_I} \left(\frac{g_z - 1 + \delta}{g_z} k \right) + h^G \right) \right. \\ & \left. + \left(\frac{P_X}{P_C} - \frac{P_{IM}}{P_C} (1-\nu_X) \left(\frac{P_{IM}}{P_C} \frac{P_C}{P_X} \varphi^X \right)^{-\mu_X} \right) x \right\} \frac{1}{g_z^{-\alpha} k^\alpha N^{1-\alpha} - \psi} \end{aligned} \tag{45}$$

Solving the Reduced Steady-State Model

Conditional on the relative price of the bundle of imported goods P_{IM}/P_C, we solve for the unknown steady-state values k, c, N, P_I/P_C, P_H/P_C, P_X/P_C and P_Y/P_C using numerical methods. In so doing, we simultaneously calibrate some key steady-state ratios of the model in order to pin down $g = h^G$, x and ψ. First, we choose the desired level of the nominal government consumption share $s_G = (P_G\, G)/(P_Y\, Y)$. Second, we calibrate the desired nominal import shares $s_{IM^C} = (P_{IM}\, IM^C)/(P_Y\, Y)$, $s_{IM^I} = (P_{IM}\, IM^I)/(P_Y\, Y)$ and $s_{IM^X} = (P_{IM}\, IM^X)/(P_Y\, Y)$ by appropriately adjusting the quasi-share parameters ν_C, ν_I and ν_X. Imposing balanced trade in steady state, the nominal export share $s_X = (S\, P_X\, X)/(P_Y\, Y)$ is then given by $s_X = s_{IM} = s_{IM^C} + s_{IM^I} + s_{IM^X}$. In addition, we choose the fixed cost in production ψ such that firms' profits are zero in steady state:

$$h + \frac{P_X}{P_H} x = \frac{1}{\varphi^H} h + \frac{1}{\varphi^X} \frac{P_X}{P_H} x + \frac{1}{\varphi^H} \psi, \tag{46}$$

where we have made use of the fact that, in steady state, the marginal costs of the two types of intermediate goods are equal to P_H/φ^H and P_X/φ^X, respectively.

Finally, we calibrate the desired nominal investment share $s_I = (P_I\, I)/(P_Y\, Y)$ by appropriately adjusting the level of the capital income tax rate τ^K.[4] The nominal consumption share $s_C = (P_C\, C)/(P_Y\, Y)$ can then be determined as a residual; that is, $s_C = 1 - s_I - s_G - (s_X - s_{IM})$.

[4]Alternatively, the investment share could be calibrated by adjusting the rate of capital depreciation δ or the capital share in production α.

As the price of imported goods is set by monopolistically competitive firms abroad, we treat the import price P_{IM} as exogenous. Hence, without loss of generality, we can normalise the relative price of imports to one; that is, we set $P_{IM}/P_C = 1$. As a result, most relative prices are equal to one as well; that is, $P_I/P_C = P_H/P_C = P_G/P_C = 1$. The relative price of export goods, P_X/P_C, however, may deviate from unity even in case when the market power in both domestic and foreign markets is assumed to be the same. Similarly, also the relative price of output, P_Y/P_C, will differ from unity.[5]

Because of $P_{IM}/P_C = P_I/P_C = 1$, the model's steady state is invariant to changes in the intratemporal substitution elasticities between domestic and imported intermediate goods, μ_C and μ_I. In contrast, the steady state, notably the relative prices P_X/P_C and P_Y/P_C, depends on the intratemporal substitution elasticity between the domestic and imported intermediate goods in the production of the exported intermediate goods, μ_X. Furthermore, the model's steady state depends on the trend growth rate g_z, the inverse of the labour-supply elasticity ζ and—via Θ—the habit parameter κ. However, while the parameters ζ and κ do influence the steady-state *level* of real variables such as labour, capital and consumption, their steady-state *ratios* are invariant to changes in those parameters. In contrast, the trend growth rate g_z also influences key steady-state ratios, including the capital-to-labour ratio. Hence, as the log-linearised model is parameterised in terms of steady-state ratios rather than steady-state levels, it is only the variation in the intratemporal substitution elasticity μ_X and the trend-growth rate g_z that would eventually require the updating of the steady-state computations at the estimation stage.

[5] The relative prices of the intermediate goods sold abroad and of output are equal to unity in case the import content of the intermediate goods sold abroad is assumed to be zero; i.e., $\nu_X = 1$.

References

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